16.2 Line Integrals

Def If f is defined on a smooth curve C given by $x = x(t)$, $y = y(t)$, $a \le t \le b$, then the line integral of f along C is

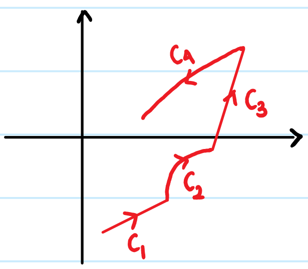
$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

In the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Then we can show that,

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$



If C is a piecewise-smooth curve ; i.e C is a union of finite number of smooth curves $C_1, \dots, C_n$ where, the initial point of C_{i+1} is the terminal point of C_i.

Then, $$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \dots + \int_{C_n} f(x,y)\,ds$$

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There are two other line integrals obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$

They are called line integrals of f along C wrt x and y

$$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i \quad ; \quad \int_C f(x,y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i$$

$$= \int_a^b f(x(t), y(t))\,x'(t)\,dt \qquad\qquad = \int_a^b f(x(t), y(t))\,y'(t)\,dt$$

It frequently happens that line integrals w.r.t to x and y occur together.

In such a case we write it as

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

- <u>Remark</u> When we are setting up a line integral, the hardest thing sometimes is to find a parametric representation for a curve whose geometric description is given.

- For instance, we often need to parametrize a line segment.

A vector representation of a line that starts at $\vec{r_0}$ and ends at $\vec{r_1}$ is given by :

$$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1} \quad , \quad 0 \le t \le 1$$

<u>Ex</u> Evaluate $\int_C y^2\,dx + x\,dy$ where

i) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$

ii) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$

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i) The parametrization of $C = C_1$ is given by

$$\vec{r}(t) = (1-t)\langle -5,-3\rangle + t\langle 0,2\rangle \quad , 0 \le t \le 1$$
$$= \langle 5t-5, 3t-3\rangle + \langle 0, 2t\rangle$$
$$= \langle 5t-5, 5t-2\rangle$$

So, $x = 5t-5$, $y = 5t-3$, $0 \le t \le 1$.

$$\int_C y^2\,dx + x\,dy = \int_0^1 (5t-3)^2 5\,dt + (5t-5)\,5\,dt$$
$$= 5\int_0^1 (25t^2 - 25t + 4)\,dt = -\frac{5}{6}$$

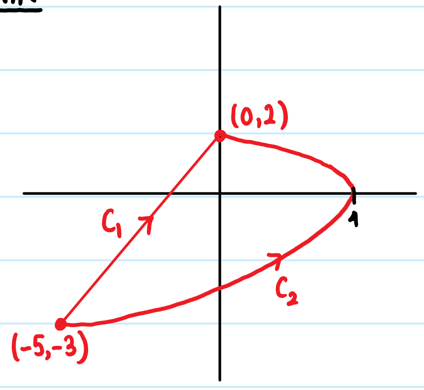
ii) Since the parabola is given as a function of y, we can take y as a parameter.
Then,

$$x = 4 - y^2 \ , \ y = y \ , \ -3 \le y \le 2$$

Then $dx = -2y\,dy$ and

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + (4-y^2)\,dy = \int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy = \frac{245}{6}$$

Rmk



The two curves have the same endpoints, but the integrals are different.

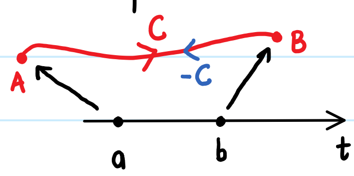
So in general, the value of the line integrals depend on the path and not only on the endpoints

Also, the answers depend on the direction or the orientation of curve.

If $-C_1$ denotes the line segment from $(0,2)$ to $(-5,-3)$, then you can show

$$\int_{-C_1} y^2 dy + x dy = \frac{5}{6}$$

- In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an orientation of the curve C,
  the positive direction of travel corresponds to increasing values of t.



$A \equiv$ initial point (i.e. when $t = a$)

$B \equiv$ terminal point (when $t = b$)

Then $-C$ means the same curve but w/ orientation reversed (opposite orientation)

So, $\displaystyle\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$; $\displaystyle\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$

However, $\displaystyle\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds$

This is because in our defn Δs_i is always positive, where as Δx_i and Δy_i change signs when we reverse orientation.

<u>Line Integrals in space</u>

Suppose that C is a smooth space curve given by the parametric equations

$x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$

or in a vector equation form $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$.

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If f is a function of 3 variables continuous on some region containing C, then we define the line integral of f along C as

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

and we can evaluate it as :

$$\int_C f(x,y,z)\,dS = \int_a^b f(x,y,z)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

<u>Remark</u> If $\vec{r}(t) = (x(t), y(t), z(t))$

then $\vec{r}\,'(t) = (x'(t), y'(t), z'(t)) \Rightarrow |\vec{r}\,'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

So, $\int_C f(\vec{r}(t))\,ds = \int_a^b f(\vec{r}(t)).|\vec{r}\,'(t)|\,dt$ [Holds for function of 2 variable as well]

When $f(x,y,z) = 1$,

$$\int_C f(x,y,z)\,dS = \int_a^b |\vec{r}\,'(t)|\,dt = L.$$

Just like in the case of function of 2-variables, we can define like integrals along C w.r.t x, y and z.

For instance, $\int_C f(x,y,z)\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta z_i = \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt$

<u>Notation</u>

$$\int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

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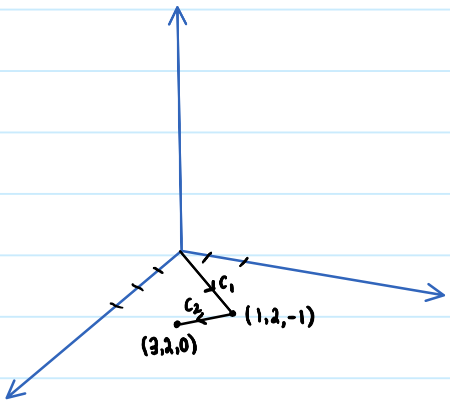
<u>Ex</u> Evaluate $\int_C x^2dx + y^2dy + z^2dz$, where C consists of line segments

C_1 from $(0,0,0)$ to $(1,2,-1)$ and C_2 consists of line segments from $(1,2,-1)$ to $(3,2,0)$

<u>Soln</u> We need to parametrize C_1 and C_2



C_1: $\vec{r}(t) = (1-t)\langle 0,0,0\rangle + t\langle 1,2,-1\rangle$

$= \langle t, 2t, -t\rangle, \quad 0 \le t \le 1$

or equivalently $x = t, \; y = 2t, \; z = -t, \quad 0 \le t \le 1$

$dx = dt, \; dy = 2dt, \; dz = -dt$

$$\text{Thus, } \int_{C_1} x^2dx + y^2dy + z^2dz = \int_0^1 t^2dt + (2t)^2\, 2dt + (-t)^2 \cdot -dt = \int_0^1 8t^2\,dt = \frac{8}{3}t^3\Big]_0^1 = \frac{8}{3}$$

Likewise along C_2,

$\vec{r}(t) = (1-t)\langle 1,2,-1\rangle + t\langle 3,2,0\rangle$

$= \langle 1-t, 2-2t, t-1\rangle + \langle 3t, 2t, 0\rangle$

$= \langle 2t+1, 2, t-1\rangle, \quad 0 \le t \le 1$

$dx = 2, \; dy = 0, \; dz = 1$

$$\int_{C_2} x^2dx + y^2dy + z^2dz = \int_0^1 (1+2t)^2 \cdot 2dt + (t-1)^2dt = \int_0^1 9t^2 + 6t + 3\,dt = \Big[3t^3 + 3t^2 + 3t\Big]_0^1 = 9$$

$$\text{Then } \int_C x^2dx + y^2dy + z^2dz = \frac{8}{3} + 9 = \frac{35}{3}.$$

<u>Line Integrals of vector fields</u>

<u>Def</u> Let $\vec{r}(t)$, $a \le t \le b$, be a smooth curve.

The unit tangent vector of curve C is given by $T(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

DEF Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $r(t)$, $a \le t \le b$.

Then the line integral of $\vec{F}$ along C is

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b F(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot\vec{T}\,ds$$

Physical Interpretation of a line integral:

Work The work W done by a force field $\vec{F}$ acting on a moving particle along a smooth curve C given by a vector function $\vec{r}(t)$, $a \le t \le b$ is given by:

$$W = \int_C \vec{F}(\vec{r}(t))\cdot\vec{T}(t)\,ds$$

- The work is the line integral w.r.t arc length of the tangential component of the force.

Ex Find the work done by the force field $\vec{F}(x,y) = \langle x, y+2\rangle$ in moving an object along an arch of the cycloid $\vec{r}(t) = (t-\sin t)\hat{\imath} + (1-\cos t)\hat{\jmath}$, $0 \le t \le 2\pi$.

Soln $\vec{F}(\vec{r}(t)) = (t-\sin t)\hat{\imath} + (1-\cos t+2)\hat{\jmath}$

$$= \langle t-\sin t, 3-\cos t\rangle$$

$$\vec{r}\,'(t) = \langle 1-\cos t, \sin t\rangle$$

Then, $W = \int_C \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi} \langle t-\sin t, 3-\cos t\rangle\cdot\langle 1-\cos t, \sin t\rangle\,dt$

$$= \int_0^{2\pi} (t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cdot\cos t)\,dt$$

$$= \int_0^{2\pi} (t - t\cos t + 2\sin t)\,dt = \left[\frac{1}{2}t^2 - (t\sin t + \cos t) - 2\cos t\right]_0^{2\pi}$$

$$= 2\pi^2.$$

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<u>Remark</u> $\int_{-C} \vec{F}.d\vec{r} = -\int_{C} \vec{F}.d\vec{r}$ [Since $\vec{T}$ is reversed]

Connection between Line Integrals of vector field and line integrals of scalar field.

Let $\vec{F} = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$
and C is a smooth curve given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$.

Then,
$$\int_C \vec{F}.d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)).\vec{r}\,'(t)$$
$$= \int_a^b (P\hat{i} + Q\hat{j} + R\hat{k}).(x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k})\,dt$$
$$= \int_a^b \left[P(x(t),y(t),z(t))x'(t) + Q(x(t),y(t),z(t)).y'(t) + R(x(t),y(t),z(t)\,z'(t)\right] dt$$
$$= \int_C P\,dx + Q\,dy + R\,dz$$

<u>DEF</u> A vector field $\vec{F}$ is called a <u>conservative vector field</u> if it is the gradient of some scalar function f
i.e. $\vec{F} = \nabla f$. In this situation f is called a potential function for $\vec{F}$.

<u>Ex</u> Suppose an electric charge Q is located at the origin.
According to Coulumb's Law, the electric force $\vec{F}(\vec{x})$ exerted by this charge on a charge q located at a point (x,y,z) w/ position vector $\vec{x} = \langle x,y,z\rangle$ is

$$\vec{F}(\vec{x}) = \frac{\varepsilon q Q}{|\vec{x}|^3}\vec{x}$$ where ε is constant.

For charges w/ same sign, $qQ > 0$ and the force is repulsive.
opposite sign, $qQ < 0$ and the force is attractive.

Let $f(x,y,z) = -\frac{\varepsilon q Q}{\sqrt{x^2+y^2+z^2}}$

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\varepsilon q Q x}{(x^2+y^2+z^2)^{3/2}}\hat{i} + \frac{\varepsilon q Q y}{(x^2+y^2+z^2)^{3/2}}\hat{j} + \frac{\varepsilon q Q z}{(x^2+y^2+z^2)^{3/2}}\hat{k}$$

$$= \vec{F}(x,y,z)$$

16.3 Fundamental Thm for Line Integrals

FTC $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

Theorem Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$.
Let f be a diff. function of two or three variables whose gradient vector ∇f is continuous on C. Then,

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

- Line integral of ∇f is the net change in f.

Pf Let us assume f is a function of 3 variables.

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t)) \right] dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)).$$

Remark: We can evaluate the line integral of a conservative vector field (the gradient vector field of a potential function f) simply by knowing the value of f at the endpoints of C.

Remark: Also true for piecewise smooth curves.

- Recall $\vec{F} = -\frac{GmM}{|\vec{x}|^3}\vec{x}$.

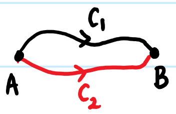
$\vec{F}$ is a conservative vector field w/ potential function $f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}$ i.e $\nabla f = \vec{F}$.

Ex Find the work done by the gravitational field $\vec{F}(x) = -\frac{GmM}{|\vec{x}|^3}\vec{x}$ in moving a particle w/ mass m from the point $(3,4,12)$ to $(2,2,0)$ along a piecewise-smooth curve C.

Solution $W = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(2,2,0) - f(3,4,12) = \frac{GmM}{\sqrt{2^2+2^2}} - \frac{GmM}{\sqrt{3^2+4^2+12^2}}$

$$= GmM\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)$$

Independence of path



Suppose C_1 and C_2 are two piecewise-smooth curves (which are called paths) that have the same initial point A and terminal point B.

We saw previously an example where $\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_2}\vec{F}\cdot d\vec{r}$

But the FTLI says, $\int_{C_1}\nabla f\cdot d\vec{r} = \int_{C_2}\nabla f\cdot d\vec{r}$ whenever ∇f is continuous.

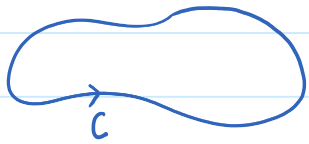
i.e. the line integral of a conservative vector field depends only on the initial & terminal pt of the curve.

Defn If $\vec{F}$ is continuous vector field w/ domain D, we say that the line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of path if $\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_2}\vec{F}\cdot d\vec{r}$ for any two path C_1, C_2 w/ same initial & terminal points.

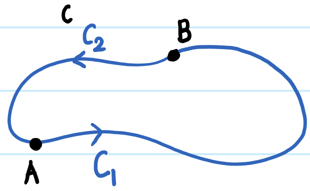
- The line integrals of conservative vector fields are independent of path.

DEFN

A curve is called closed if it's terminal point coincides w/ the initial point. $(\vec{r}(a) = \vec{r}(b))$.



If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, and C be any closed path in D, we can pick any two points A and B on C and treat C as a composition of path C_1 from A to B and C_2 from B to A.



Then, $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r} = 0$ since C_1 and $-C_2$ have same initial and terminal points.

Now on the other hand, if $\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed path C in D,

take two paths C_1, C_2 from A to B and define C to be the curve C_1 followed by $-C_2$.

Then,

$$0 = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} \Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}.$$

So we have shown :

<u>Theorem</u> $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C in D.

<u>Physical interpretation</u> : Work done by conservative force field as it moves an object around a closed path is 0.